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**Spectral Graph Theory and Applications 2016
on occasion of Dragoš Cvetković's 75th birthday**

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**Star complement technique
and the eigenbasis of -2 in signed line graphs**

Based on joint researches with E.M Li Marzi and S.K. Simić.

Outline

- 1 Preliminaries
 - Basic notions on Signed Graphs
 - Matrices of Signed Graphs
- 2 Signed Line Graphs
 - Signed graphs, Bi-directed graphs and Line graphs
 - A -polynomials of signed line graph
- 3 The eigenspace of -2
 - Star Set and Star Complement
 - Main result

Signed Graphs

A *signed graph* Γ is an ordered pair (G, σ) , where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \rightarrow \{+, -\}$ is the signature function (or sign mapping) on the edges of G .

Signed Graphs

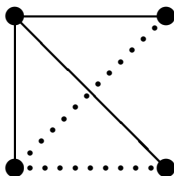
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Example of a signed graph.

positive edges = solid lines;

negative edges = dotted lines.

More on Signed Graphs

Signed graphs were first introduced by Harary to handle a problem in social psychology (Cartwright and Harary, 1956). Recently, signed graphs have been considered in the study of complex networks, and Godsil et al. showed that negative edges are useful for perfect state transfer in quantum computing.

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In most applications of signed graphs there is a recurring property that naturally arises:

Definition

A signed graph is said to be balanced if and only if all its cycles are positive.

Balance

The first characterization of balance is due to Harary:

Theorem (Harary, 1953)

A signed graph is balanced iff its vertex set can be divided into two sets (either of which may be empty), so that each edge between the sets is negative and each edge within a set is positive.

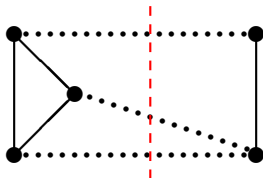
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The above theorem shows that balancedness is a generalization of the ordinary bipartiteness in (unsigned) graphs.



A balanced signed graph.
The dashed line separates
the two clusters.

Signature Switching

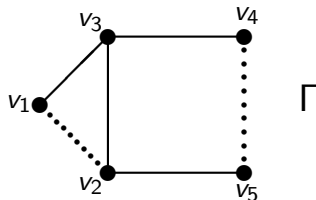
Definition

Let $\Gamma = (G, \sigma)$ be a signed graph and $U \subseteq V(G)$. The signed graph Γ^U obtained by changing the edges signs in the cut $[U; U^c]$ is a (sign) switching of Γ . We also say that the signatures of Γ^U and Γ are equivalent.

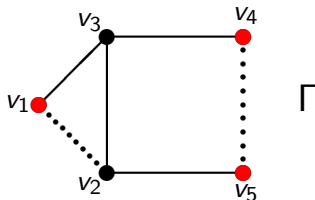
The signature switching preserves the set of the positive cycles.

In general, we say that two signed graphs are *switching isomorphic* if their underlying graphs are isomorphic and the signatures are switching equivalent. The set of signed graphs switching isomorphic to Γ is the *switching isomorphism class* of Γ , written $[\Gamma]$.

Example of switching equivalent graphs

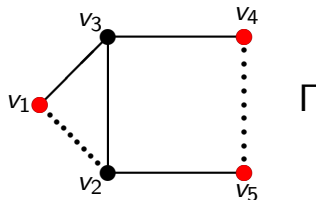


Example of switching equivalent graphs

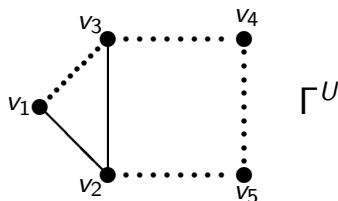


Let $U = \{v_1, v_4, v_5\}$.

Example of switching equivalent graphs



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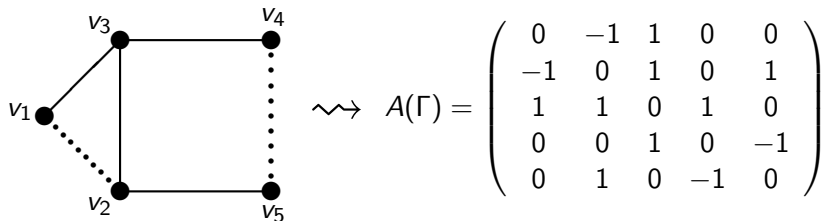


Note that the switching preserves the signs of all cycles!

Adjacency matrix of Signed Graphs

The adjacency matrix is defined as $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i v_j), & \text{if } v_i \sim v_j; \\ 0, & \text{if } v_i \not\sim v_j. \end{cases}$$

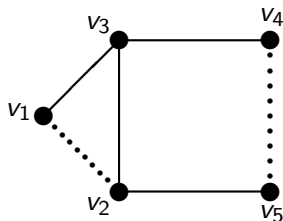


Laplacian of Signed Graphs

The Laplacian matrix of $\Gamma = (G, \sigma)$ is defined as

$$L(\Gamma) = D(G) - A(\Gamma) = (l_{ij})$$

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j; \\ -\sigma(v_i v_j), & \text{if } i \neq j. \end{cases}$$



$$\rightsquigarrow L(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

Switching and signature similarity

What happens if we pick two signed graphs from the same switching class?

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Switching has a matrix counterpart. In fact, Let Γ and $\Gamma' = \Gamma^U$ be two switching equivalent graphs.

Consider the matrix $S_U = \text{diag}(s_1, s_2, \dots, s_n)$ such that

$$s_i = \begin{cases} +1, & i \in U \\ -1, & i \in \Gamma \setminus U \end{cases}$$

S_U is called a *signature matrix* (or *state matrix*).

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It is easy to check that

$$A(\Gamma^U) = S_U A(\Gamma) S_U \quad \text{and} \quad L(\Gamma^U) = S_U L(\Gamma) S_U.$$

Hence, signed graphs from the same switching class share similar graph matrices, or switching isomorphic graphs are cospectral.

Balance and signature switching

The following theorem is pretty evident:

Theorem

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Proof. If the graph is balanced then it admits a bipartition in a 2-clusters, so we can switch all the negative edges to positive edges. On the other hand, if the signed graph is switching equivalent to the all positive signature then all cycles are balanced and then the whole graph is balanced as well. \square

A signed graph that is switching equivalent to the all negative signature is said to be *antibalanced*.

(signless) Laplacian spectral theory of unsigned graphs

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positive edges

$$\sigma(e) = +1 \text{ for all } e \in E(G)$$

$$A(\Gamma) = A(G)$$

$$L(\Gamma, +) = L(G)$$

we have the usual Laplacian matrix of G .

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The Laplacian Theory of signed Graphs can be seen as a generalization of those of unsigned graphs.

Bi-directed graphs and signed graphs

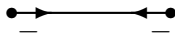
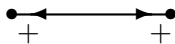
An oriented signed graph is an ordered pair $\Gamma_\eta = (\Gamma, \eta)$, where

$$\eta : V(G) \times E(G) \rightarrow \{-1, +1, 0\}$$

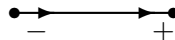
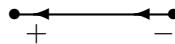
satisfies the following three conditions:

- (i) $\eta(u, vw) = 0$ whenever $u \neq v, w$;
- (ii) $\eta(v, vw) = +1$ (or -1) if an arrow at v is going into (resp. out of) v ;
- (iii) $\eta(v, vw)\eta(w, vw) = -\sigma(vw)$.

unoriented edges:



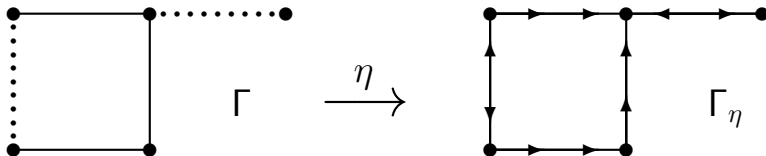
oriented edges:



Bidirected edges

Bi-directed graphs and signed graphs

So we have that positive edges are oriented edges, while negative corresponds to unoriented. Thus each bi-directed graph is a signed graph. The converse is also true, but then one arrow (at any end) can be taken arbitrarily, while not the other arrow (in view of (iii)).



Incidence matrix

The *incidence matrix* of Γ_η is the matrix $B_\eta = (b_{ij})$, whose rows correspond to vertices and columns to edges of G , such that

$$b_{ij} = \eta(v_i, e_j),$$

with $v_i \in V(G)$ and $e_j \in E(G)$.

It is not difficult to see that

$$B_\eta B_\eta^T = D(G) - A(\Gamma) = L(\Gamma),$$

where $D(G)$ is the diagonal matrix of vertex degrees of G . So $L(\Gamma)$ is positive-semidefinite.

Note any choice for η leads to the same matrix $L(\Gamma)$!

Signed line graphs and incidence matrix

On the other hand,

$$B_{\eta}^T B_{\eta} = 2I + A(\mathcal{L}(\Gamma_{\eta})),$$

where $\mathcal{L}(\Gamma_{\eta})$ is a signed graph whose underlying graph is $\mathcal{L}(G)$.

The signed line graph of $\Gamma = (G, \sigma)$ is the signed graph $(\mathcal{L}(G), \sigma_{\eta}^{\mathcal{L}})$, where $\mathcal{L}(G)$ is the (usual) line graph and

$$\sigma_{\eta}^{\mathcal{L}}(e_i e_j) = \begin{cases} b_{ki}^{\eta} b_{kj}^{\eta} & \text{if } e_i \text{ is incident } e_j \text{ at } v_k; \\ 0 & \text{otherwise.} \end{cases}$$

Assigning a different orientation η' will lead to a different $\mathcal{L}(\Gamma_{\eta'})$, however we have that $\mathcal{L}(\Gamma_{\eta})$ and $\mathcal{L}(\Gamma_{\eta'})$ are switching equivalent!

A well-known formula

So the signed line graph $\mathcal{L}(\Gamma)$ is uniquely defined up to switching isomorphisms. On the other hand,

$$B_\eta B_\eta^T = L(\Gamma), \quad \text{and} \quad B_\eta^T B_\eta = \mathcal{L}(\Gamma)$$

share the same non-zero eigenvalues, therefore:

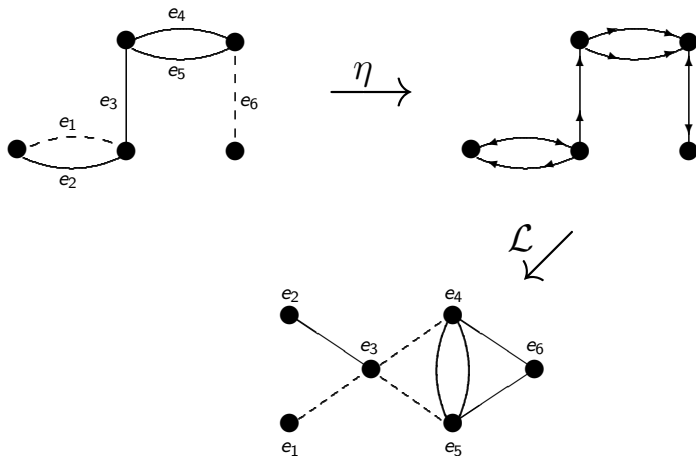
Theorem

Let Γ be a signed graph of order n and size m , and let $\phi(\Gamma)$ and $\psi(\Gamma)$ be its adjacency and Laplacian characteristic polynomials, respectively. Then it holds

$$\phi(\mathcal{L}(\Gamma), x) = (x + 2)^{m-n} \psi(\Gamma, x + 2).$$

An example

A signed multigraph and its corresponding signed line graph



Remark

The signed line graph operator just defined can give rise to signed line graphs with all edges positive (so an “unsigned” line graph).

In fact, $\mathcal{L}(\Gamma_\eta) = \mathcal{L}(G)_+$ if:

- (a) Γ_η connected with at least 3 vertices;
- (b) for any vertex v in Γ_η , $\eta(v, e)$ is constant for all edges e incident to v , with possible exception if v is a vertex of degree 2 in some 2-cycle.

The latter bi-directed graph Γ_η can be obtained if and only if Γ is equivalent to G_- , with possible exception within hanging 2-cycles. So, Hoffman’s generalized line graphs can be obtained from their signed root graphs, where petals become unbalanced 2-cycles.

Notation

Let $\Gamma = G_\sigma$ be a signed graph and λ be an (adjacency) eigenvalue of multiplicity k . For any induced subgraph H of Γ , the signs of edges of H are inherit from Γ ,

- $X \subseteq V(G)$ is a *star set* of G if $|X| = k$ and λ is not an eigenvalue of $H = G - X$;
- H is a *star complement* of G with respect to λ .

We focus our attention to the case $\lambda = -2$ and Γ is a signed line graph.

General facts on Star Complements

These facts can be found in the book of Cvektović, Rowlinson and Simić on the graphs with least eigenvalue -2 (the light green book). They hold for Hermitian Matrices, so they can be applied to weighted graphs and signed graphs, as well:

- (i) *Any weighted graph has a star complement (and a star set) for any eigenvalue.*
- (ii) *If $\lambda \neq 0$ then each vertex from a star set has a neighbour in the corresponding star complement.*
- (iii) *Every connected weighted graph has a connected star complement for any eigenvalue.*

General facts on Star Complements (cont.)

- (iv) A basis for some $\lambda \in \text{Sp}(G_\sigma)$ with respect to star complement H_σ can be constructed as follows: for each $v \in X$ we consider the signed subgraph $(H + v)_\sigma$ and construct a unique (up to scalar factor) eigenvector \mathbf{y}_v ; by extending the latter vector with zero entries corresponding to vertices from $X \setminus \{v\}$ we obtain a vector \mathbf{x}_v which is an eigenvector for G_σ .
- (v) *For the least eigenvalue, every vertex is either a downer or neutral (it follows from the Interlacing theorem).*

Now we look for the star complements in a signed line graph by means of the corresponding subgraphs in the root graph.

This result is the signed variant of a famous result (Doob and Cvektović?):

Lemma

If Γ is a connected signed graph on m edges then

$$|\phi(\mathcal{L}(\Gamma), -2)| = \begin{cases} m + 1 & \text{if } \Gamma \text{ is a tree,} \\ 4 & \text{if } \Gamma \text{ is an unbalanced unicyclic graph,} \\ 0 & \text{otherwise.} \end{cases}$$

An immediate consequence of the above lemma:

Corollary

Let Γ be a connected signed graph. Then -2 is the least eigenvalue of $\mathcal{L}(\Gamma)$ if and only if Γ contains as a signed subgraph at least one balanced cycle, or at least two unbalanced cycles.

The signed foundation

From now on Γ is a connected signed graph and $\Gamma_{\mathcal{L}}$ is the corresponding signed line graph. We assume also that -2 is an eigenvalue of multiplicity k .

We consider a star complement of -2 in $\Gamma_{\mathcal{L}}$, such a star complement is obtained from edges of Γ . Let Φ be the corresponding “line star complement” in Γ . We can take Φ (the signed foundation) to be connected.

Therefore, Φ is:

- a tree;
- an unbalanced unicyclic graph (2-cycles are allowed!).

Core of $\Phi + e$

When we add an edge e to Φ we get a subgraph of Γ for which the signed line graph has -2 as an eigenvalue. The -2 -eigenvector might have some 0-components.

Let Θ be the subgraph of $\mathcal{L}(\Phi + e)$ induced by the vertices (edges in Γ) with non-zero entries.

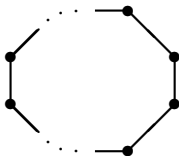
The following facts hold for the core Θ :

- (i) *The edge e belongs to Θ .*
- (ii) *Θ is connected.*
- (iii) *The edge e belongs to some cycle of Θ .*
- (iv) *There are no pendant edges in Θ .*
- (v) *2-cycles if exist must be unbalanced.*

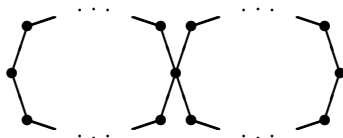
General facts on the cores

From the above facts, the core Θ is one of the following graphs:

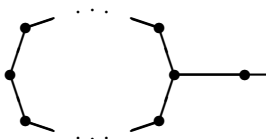
- a balanced cycle;
- a double unbalanced dumbbell;
- a double unbalanced infinite graph.



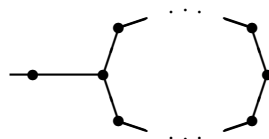
(balanced cycle)



(unbalanced cycles)



(unbalanced cycle)



(unbalanced cycle)

Eigenvector for $\lambda = -2$; Balanced cycle

In the following theorem the -2 -eigenvector is described.

Theorem (Balanced cycle)

Let Θ be a balanced cycle and Θ_L its line (signed) graph. Then the vector $\mathbf{a} = (a_0, a_1, \dots, a_{q-1})^\top$, where

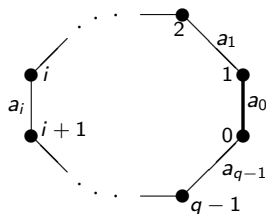
$$a_i = (-1)^i \left[\prod_{s=1}^i \nu(s) \right] a_0 \quad (i = 0, 1, \dots, q-1),$$

where $\nu(s) = \sigma_L(e_{s-1}e_s)$, is an eigenvector of Θ_L for -2 . Moreover, it can be extended to a (-2) -eigenvector of Γ_L by inserting zeros at remaining entries.

The eigenvector for $\lambda = -2$:

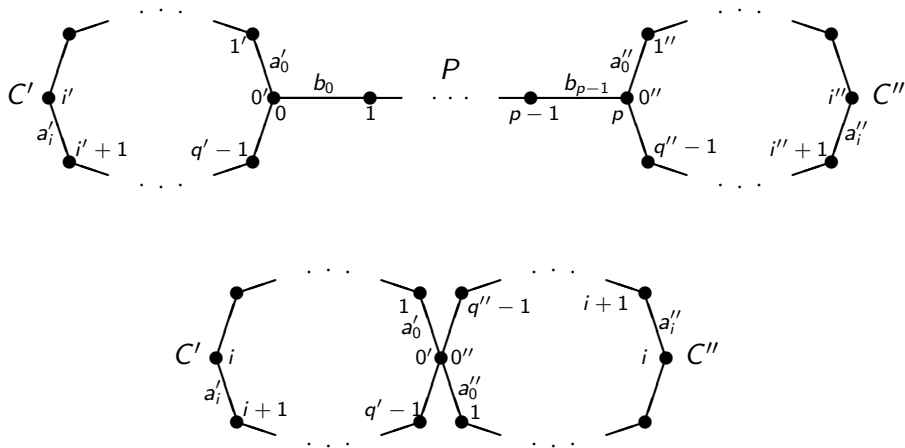
$$a_i = (-1)^i \left[\prod_{s=1}^i \nu(s) \right] a_0 \quad (i = 0, 1, \dots, q-1),$$

where $\nu(j) = \sigma_L(e_{j-1}e_j)$.



In the case of unbalanced cores we have a similar result.

Eigenvector for $\lambda = -2$; Unbalanced cores



Thanks God, *it's over*. Those who are interested on the details of the eigenvector components are referred to

[FB, EM Li Marzi, SK Simić, Signed line graphs with least eigenvalue -2 : The star complement technique, Discrete Applied Math. 207 (2016), 29–38.]

Thank you!!